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FIXED POINT MEROMORPHIC FUNCTION WITH DIFFERENCE POLYNOMIALS

JAYARAMA H. R*, HARISH H, CHAITHRA C. N, AND NAVEENKUMAR S. H

ABSTRACT. In this paper, we let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$ and $c_1, ..., c_m$ be complex constants satisfying that at least one of them is non-zero. The authors establish fixed points about the difference polynomials $\Phi(z) = f(z+c_1)f(z+c_2)...f(z+c_m) - a(f(z))^n$. These results extend the related results obtained by Zhaojun Wu1 and Jia Wu [18].

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1. Introduction

In this paper, we study the value distribution of difference polynomials of meromorphic functions by using the Nevanlinna theory. Therefore, we use the basic notations of the Nevanlinna theory and assume that the reader knows these notation (see [14] [20] [23]). Let $f(z)$ be a non-constant meromorphic function defined in the complex plane $\mathbb{C}$, if $a \in \mathbb{C}$. Nevanlinna defined the following functions.

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} log^+|f(re^{i\theta})|d\theta,$$

$$m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} log^+\frac{1}{|f(re^{i\theta})-a|}d\theta,$$

$$N(r,f) = \frac{1}{2\pi}\int_0^{r} \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)logr,$$

$$N\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{r} \frac{n\left(t, \frac{1}{f-a}\right) - n\left(0, \frac{1}{f-a}\right)}{t}dt + n\left(0, \frac{1}{f-a}\right)logr,$$

$$T(r,f) = m(r,f) + N(r,f),$$

$$T\left(r, \frac{1}{f-a}\right) = m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right),$$

where $T(r,f)$ is called the characteristic function of $f(z)$, $log^+x = \max\{logx, 0\}$ $(x > 0)$, $n(t,f)$ and $n\left(t, \frac{1}{f-a}\right)$ denote the number of poles of $f(z)$ and the number of zeros of $f(z)-a$ in the disc $|z| \le t$, counting multiplicities.

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We use $S(r, f)$ to denote any quantity of $S(r, f) = o\{T(r, f)\}$ $(r \longrightarrow \infty)$, possibly outside a set E with finite logarithmic measure.

The order of $f(z)$ is denoted by $\sigma(f)$. For any $a \in \mathbb{C}$, we use the notations $\sigma(f)$ and $\lambda(f, a)$ denote the order of $f(z)$ and the exponent of convergence of zeros of $f(z)-a$, respectively. If $a = 0$, we denote $\lambda(f, 0) = \lambda(f)$. A point $z \in \mathbb{C}$ is called a fixed point of $f(z)$, if $f(z) = z$. There are a considerable number of results on the fixed points for meromorphic functions in the plane. We refer the reader to Chuang and Yang [10]. According to Chen and Shon [6], the notation $\tau(f)$ is used to represent the exponent of convergence of fixed points of the function f, which is defined as

$$\tau(f) = 1 - \lim_{r\to\infty} \sup \frac{logN\left(r, \dfrac{1}{f-z}\right)}{logr},$$

In 1993, Lahiri [15] proved the following theorem

Theorem 1.1. *Let f be a transcendental meromorphic function in the plane. Suppose that there exists $a \in \mathbb{C}$ with $\delta(a, f) > 0$ and $\delta(\infty, f) = 1$. Then f has infinitely many fixed points.*

In 2004, , Yi and Yang [20] have proved the following theorem.

Theorem 1.2. *Let f be a transcendental meromorphic function in $\mathbb{C}$ with a positive order. If f has two distinct Borel exceptional values, say a_1 and a_2, then the order of f is a positive integer or ∞ and $\sigma(f) = \mu(f)$, $\delta(a_1, f) = \delta(a_2, f) = 1$.*

When the order of f is less than 1, Chen and Shon [6] have proved the following.

Theorem 1.3. *Let f be a transcendental meromorphic function of order of growth $\sigma(f) \leq 1$. Suppose that f satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then Δf has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(\Delta f) = \sigma(f)$.*

In 2014, Hongyan Xu and Zhaojun Wu [19] have proved the following.

Theorem 1.4. *Let f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$ and $a \in \mathbb{C}$. Suppose that f satisfies $\lambda(1/f) < \sigma(f)$ and $\lambda(f, a) < \sigma(f)$. Then Δf has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(\Delta f) = \sigma(f)$.*

For the existence on the fixed points of differences, Cui and Yang [17] have proved the following theorems.

Theorem 1.5. *Let f be a function transcendental and meromorphic in the plane with the order $\sigma(f) = 1$. If f has finitely many poles and infinitely many zeros with exponent of convergence of zeros $\lambda(f) \neq 1$, then Δf has infinitely many zeros and fixed points.*

Theorem 1.6. *Let f be a function transcendental and meromorphic in the plane with the order $\sigma(f) = 1$ $max\{\lambda(f), \lambda\frac{1}{f}\} = 1$. If f has infinitely many zeros, then Δf has infinitely many zeros and fixed points.*

The conditions of Theorems 1.5 and 1.6 imply that 0, ∞ are Borel exceptional values. In cased of when ∞ and $d \in \mathbb{C}$ are Borel exceptional values of f, Chen [5] obtains the following theorem.

Theorem 1.7. *Let f be a finite order meromorphic function such that $\lambda(\frac{1}{f}) < \sigma(f)$, and let $c \in \mathbb{C}\backslash\{0\}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $f(z)$ has a Borel exceptional value $d \in \mathbb{C}$, then $\tau(\Delta_c f) = \sigma(f)$.*

In 2016, Chen and Zhang [21] have obtained the following result.

Theorem 1.8. *Let f be a finite order meromorphic function, and let $c \in \mathbb{C}\backslash\{0\}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $f(z)$ has two Borel exceptional values, then $\tau(\Delta_c f) = \sigma(f)$.*

If the order of f is not a positive integer, Jia Wu and Zhaojun Wu1 [18] have obtained the following result.

Theorem 1.9. *Let f be a transcendental meromorphic function of finite order in the plane. Suppose that $c \in \mathbb{C}\backslash\{0\}$ such that $\Delta_c f \not\equiv 0$. If there is $a \in \mathbb{C}$ with $\delta(a,f) > 0$ and $\delta(\infty, f) = 1$, then $\Delta_c f$ have infinitely many fixed points and $\tau(\Delta_c f) = \sigma(f)$.*

Theorem 1.10. *Let f be a transcendental meromorphic function of finite order in the plane. Suppose that $c \in \mathbb{C}\backslash\{0\}$ such that $\Delta_c f \not\equiv 0$. If $\delta(\infty, f) = 1$, $\delta(0,f) = 1$, then*

$$T(r, \Delta_c f) \sim T(r,f) \sim N\left(r, \frac{1}{\Delta_c f - z}\right),$$

as $r \longrightarrow \infty$, $r \notin E$, where E is a possible exception set of r with finite logarithmic measure.

For c-shift difference polynomial of meromorphic functions and its certain properties, we refer to the article [[4]]. For recent developments in difference polynomials and different aspects of it, we refer to the articles [[1],[3],[16],[2]].

Let f be a meromorphic function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Nevanlinnas deficiency of f with respect to a is defined by

$$\delta(a,f) = 1 - \lim_{r\to\infty} \sup \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

If $a = \infty$, then one should replace $N\left(r, \frac{1}{f-a}\right)$ in the above formula by $N(r,f)$. If $\delta(a,f) > 0$, then a is called a Nevanlinna deficiency value of f.

Let f be a transcendental meromorphic function and $a \in \mathbb{C}\{0\}$, $m, n \in \mathbb{C}$ and $c_1, c_2, ..., c_m$ be complex constants satisfying that at least one of them is non-zero. Zheng and Chen [24] define and investigate the value distribution of difference polynomials

$$(1) \qquad \Phi(z) = f(z+c_1)f(z+c_2)...f(z+c_m) - a(f(z))^n.$$

In this article we generalize Theorems 1.9 and 1.10 to the case of difference polynomials defined above.

Theorem 1.11. *Let f be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plane. Let $c_1, c_2, ..., c_m$, $(m \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $\Phi(z) \not\equiv 0$. If there is $a \in \mathbb{C}$ with f satisfying $\delta(\infty, f) = 1$, $m = n$ and a is a Nevanlinna deficiency value of f, then $\Phi(z)$ has infinitely many fixed points and $\tau(\Phi(z)) = \sigma(f)$.*

Theorem 1.12. *Let f be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plane. Let $c_1, c_2, ..., c_m$, $(m \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $\Phi(z) \not\equiv 0$. If there is $a \in \mathbb{C}$ with f satisfying $\delta(\infty, f) = 1$, $m = n$ and $\delta(0, f) = 1$, then*

$$T(r, \Phi(z)) \sim nT(r, f) \sim N\left(r, \frac{1}{\Phi(z) - z}\right),$$

as $r \longrightarrow \infty$, $r \notin E$, where E is a possible exception set of r with finite logarithmic measure.

If $m = n = 1$ and $a = -1$, then $\Phi(z) = f(z + c) + f(z)$. We define

$$\nabla_c f(z) = f(z + c) + f(z).$$

Then, we can get the following Corollary from Theorems 1.11 and 1.12.

Corollary 1.13. *Let f be a transcendental meromorphic function of finite order $\sigma(f)$ in the plane. Suppose that $c \in \mathbb{C}\{0\}$ such that $\nabla_c f(z) \not\equiv 0$. If there is $a \in \mathbb{C}$ with $\delta(a, f) > 0$ and $\delta(\infty, f) = 1$, then $\nabla_c f$ have infinitely many fixed points and $\tau(\nabla_c f) = \sigma(f)$.*

Corollary 1.14. *Let f be a transcendental meromorphic function of finite order $\sigma(f)$ in the plane. Suppose that $c \in \mathbb{C}\{0\}$ such that $\nabla_c f \not\equiv 0$. If $\delta(\infty, f) = 1$, $\delta(0, f) = 1$, then*

$$T(r, \nabla_c f) \sim T(r, f) \sim N\left(r, \frac{1}{\nabla_c f - z}\right),$$

as $r \longrightarrow \infty$, $r \notin E$, where E is a possible exception set of r with finite logarithmic measure.

Example 1.1. *Let $f(z) = e^z$ and $c_1, c_2, ..., c_m = 1$ and $a = -1$. Then, for $m = n \in \mathbb{C}$ $\Phi(z) = f(z + c_1)f(z + c_2)...f(z + c_m) - a(f(z))^n = (e^n - 1)e^{nz}$. Obviously, we can get $\delta(0, f) = \delta(\infty, f) = 1$ and $\Phi(z)$ has infinitely many fixed points and $\tau(\Phi(z)) = \sigma(f)$.*

$$T(r, \Phi(z)) \sim nT(r, f) \sim N\left(r, \frac{1}{\Phi(z) - z}\right),$$

as $r \longrightarrow \infty$, And above all, $\Phi(z) = (e^n - 1)e^{nz} \neq 0$. Therefore, the condition $z \neq 0$ in Theorems 1.11 and 1.12 is necessary.

2. Some Lemmas

In this section, we state some lemmas which will be needed in the sequel.

Lemma 2.1. [13] *Let $f(z)$ be a transcendental meromorphic function of finite order, then*

$$m\left(r, \frac{f(z + c)}{f}\right) = S(r, f).$$

Lemma 2.2. [9] *Let $f(z)$ be a finite order meromorphic function, then, for each $k \in \mathbb{N}$, $\sigma(\Delta_c^k f) \leq \sigma(f)$.*

Lemma 2.3. [12] *Let f be a transcendental meromorphic function of finite order. Then for any positive integer n, we have*

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = S(r, f).$$

Lemma 2.4. [8][22] *Let f be a transcendental meromorphic function of finite order. Then*

$$N(r, f(z+c)) = N(r, f) + S(r, f),$$

$$T(r, f(z+c)) = T(r, f) + S(r, f),$$

where $S(r, f) = o(T(r, f))$ $(r \longrightarrow \infty)$, possibly outside a set E of r with finite logarithmic measure.

Lemma 2.5. [7] [11] *Suppose that $f(z)$ is a transcendental meromorphic function in the complex plane and $P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n$, where $a_0 (\not\equiv 0)$, $a_1, ..., a_n$ are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f),$$

Lemma 2.6. [7] [11] *Let $F(r)$ and $G(r)$ be monotone increasing functions such that $F(r) \leq G(r)$ outside of exceptional set E that is of finite logarithmic measure. Then for any $\alpha > 0$, there exists $r_0 > 1$ such that $F(r) \leq G(\alpha r)$ for all $r > r_0$.*

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plane. Let $c_1, c_2, ..., c_m$, $(m \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $\Phi(z) \not\equiv 0$, and $\delta(0, f) > 0$, if $m = n$. Then $\Phi(z)$ is a transcendental and meromorphic function of finite order.*

Proof. Since $\delta(0, f) > 0$, from Lemma 2.2, we know that $\sigma(\Phi(z)) \leq \sigma(f) < +\infty$. If $\Phi(z)$ is not a transcendental meromorphic function, then there is a rational $Q(z)$ such that $Q(z)\Phi(z) \equiv 1$, i.e.,

$$\begin{aligned} \frac{1}{f^n} &\equiv Q(z)\frac{\Phi(z)}{f^n} \\ &\equiv Q(z)\left(\left(\frac{f(z+c_1)}{f}\right).\left(\frac{f(z+c_2)}{f}\right)...\left(\frac{f(z+c_m)}{f}\right) - a\left(\frac{f(z)}{f}\right)^n\right) \end{aligned}$$

Applying Lemma 2.1 and noting that $f(z)$ is transcendental, we can get

$$m\left(r, \frac{1}{f^n}\right) \leq m(r, Q(z)) + m\left(r, \frac{\Phi(z)}{f^n}\right) = S(r, f).$$

Therefore

$$\begin{aligned} m\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{f^n}\right) &\leq N\left(r, \frac{1}{f^n}\right) + S(r, f) \\ &\leq nN\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$

Applying Lemma 2.5 and the first fundamental theorem of Nevanlinna theory, we can get

$$nT(r,f) \leq nN\left(r, \frac{1}{f}\right) + S(r,f).$$

This contradicts with $\delta(0,f) > 0$. Thus $\Phi(z)$ is a transcendental and meromorphic function of finite order. □

Lemma 2.8. *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plane. Let $c_1, c_2, ..., c_m$, $(m \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $\Phi(z) \not\equiv 0$ and $\delta(0,f) > 0$. If $m = n$, then*

$$nT(r,f) \leq nN\left(r, \frac{1}{f}\right) + 4nN(r,f) + N\left(r, \frac{1}{\Phi(z) - z}\right) + S(r,f)$$

Proof. By Lemma 2.7, we know that $\Phi(z)$ is a transcendental meromorphic function,
then there is $\eta \in \mathbb{C}\{0\}$ such that $z\Delta_\eta\Phi(z) - \Phi(z) \not\equiv 0$.
Noticing

$$\frac{1}{f^n} = \frac{\Phi(z)}{zf^n} - \frac{z\Delta_\eta\Phi(z) - \Phi(z)}{zf^n}\frac{\Phi(z) - z}{z\Delta_\eta\Phi(z) - \Phi(z)}, \tag{2}$$

then

$$\begin{aligned} m\left(r, \frac{1}{f^n}\right) \leq & m\left(r, \frac{\Phi(z)}{zf^n}\right) + m\left(r, \frac{z\Delta_\eta\Phi(z) - \Phi(z)}{zf^n}\right) \\ & + m\left(r, \frac{\Phi(z) - z}{z\Delta_\eta\Phi(z) - \Phi(z)}\right) + O(1) \\ \leq & 2m\left(r, \frac{\Phi(z)}{f^n}\right) + m\left(r, \frac{\Delta_\eta\Phi(z)}{f^n}\right) \\ & + m\left(r, \frac{\Phi(z) - z}{z\Delta_\eta\Phi(z) - \Phi(z)}\right) + O(logr). \end{aligned} \tag{3}$$

Applying the first fundamental theorem, we get

$$m\left(r, \frac{1}{f^n}\right) = nT(r,f) - nN\left(r, \frac{1}{f}\right) + O(1). \tag{4}$$

(5)

$$\begin{aligned} m\left(r, \frac{\Phi(z) - z}{z\Delta_\eta\Phi(z) - \Phi(z)}\right) = & m\left(r, \frac{z\Delta_\eta\Phi(z) - \Phi(z)}{\Phi(z) - z}\right) + N\left(r, \frac{z\Delta_\eta\Phi(z) - \Phi(z)}{\Phi(z) - z}\right) \\ & - N\left(r, \frac{\Phi(z) - z}{z\Delta_\eta\Phi(z) - \Phi(z)}\right) + O(1) \\ \leq & m\left(r, \frac{z\Delta_\eta\Phi(z) - \Phi(z)}{\Phi(z) - z}\right) + N\left(r, \frac{z\Delta_\eta\Phi(z) - \Phi(z)}{\Phi(z) - z}\right) \\ & + O(1). \end{aligned}$$

Combining (2)-(5) we have

(6)

$$\begin{aligned}
nT(r,f) \leq & nN\left(r,\frac{1}{f}\right)+2m\left(r,\frac{\Phi(z)}{f^n}\right)+m\left(r,\frac{\Delta_\eta\Phi(z)}{f^n}\right)\\
&+m\left(r,\frac{z\Delta_\eta\Phi(z)-\Phi(z)}{\Phi(z)-z}\right)\\
&+N\left(r,\frac{z\Delta_\eta\Phi(z)-\Phi(z)}{\Phi(z)-z}\right)+O(logr)\\
\leq & nN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Phi(z)-z}\right)+N(r,z\Delta_\eta\Phi(z)-\Phi(z))\\
&+2m\left(r,\frac{\Phi(z)}{f^n}\right)+m\left(r,\frac{\Delta_\eta\Phi(z)}{f^n}\right)+m\left(r,\frac{z\Delta_\eta\Phi(z)-\Phi(z)}{\Phi(z)-z}\right)\\
&+O(logr).
\end{aligned}$$

Since

$$\begin{aligned}
\Delta_\eta\Phi(z) = & \Delta_\eta(f(z+c_1)f(z+c_2)...f(z+c_m)-a(f(z)^n)\\
= & f(z+c_1+\eta)f(z+c_2+\eta)...f(z+c_m+\eta)\\
&-a((f(z+c+\eta))^n-(f(z))^n)\\
&-f(z+c_1)f(z+c_2)...f(z+c_m)\\
\Delta_\eta(\Phi(z)-z) = & f(z+c_1+\eta)f(z+c_2+\eta)...f(z+c_m+\eta)\\
&-a((f(z+c+\eta))^n-(f(z))^n)\\
&-f(z+c_1)f(z+c_2)...f(z+c_m)-(z+\eta).
\end{aligned}$$

then, we can get

$$\begin{aligned}
z\Delta_\eta\Phi(z)-\Phi(z) = & z(f(z+c_1+\eta)f(z+c_2+\eta)...f(z+c_m+\eta)\\
&-a((f(z+c+\eta))^n-(f(z))^n))\\
&-(z+1)f(z+c_1)f(z+c_2)...f(z+c_m)\\
&-(z+\eta)+a(f(z))^n.\\
z\Delta_\eta(\Phi(z)-z)-(\Phi(z)-z) = & z(f(z+c_1+\eta)f(z+c_2+\eta)...f(z+c_m+\eta)\\
&-a((f(z+c+\eta))^n-(f(z))^n))\\
&-(z+1)f(z+c_1)f(z+c_2)...f(z+c_m)\\
&+a(f(z))^n-z((z+\eta)-1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{z\Delta_\eta\Phi(z)-\Phi(z)}{\Phi(z)-z} = & \frac{z\Delta_\eta(\Phi(z)-z)-(\Phi(z)-z)}{\Phi(z)-z}\\
= & \frac{z\Delta_\eta(\Phi(z)-z)}{\Phi(z)-z}-1
\end{aligned} \tag{7}$$

(8)

$$\begin{aligned}
N(r,z\Delta_\eta\Phi(z)-\Phi(z)) \leq & N(r,f(z+c_1+\eta)f(z+c_2+\eta)...f(z+c_m+\eta))\\
&+N(r,(f(z+c+\eta))^n-(f(z))^n))\\
&+N(r,f(z+c_1)f(z+c_2)...f(z+c_m))+N(r,(f(z))^n)
\end{aligned}$$

Thus from Lemma 2.4 and (8), we deduce

$$N(r, z\Delta_\eta\Phi(z) - \Phi(z)) \leq 4nN(r, f(z)) + O(logr) \tag{9}$$

By Lemmas 2.2 and 2.7 we know that $\Phi(z) - z$ is a transcendental meromorphic function of finite order. It follows from Lemma 2.3 and (7) that

$$\begin{aligned} m\left(r, \frac{\Phi(z)}{f^n}\right) &= S(r,f) \\ m\left(r, \frac{\Delta_\eta\Phi(z)}{f^n}\right) &= S(r,f) \\ m\left(r, \frac{z\Delta_\eta\Phi(z) - \Phi(z)}{\Phi(z) - z}\right) &= S(r,f). \end{aligned} \tag{10}$$

From (6) and (9)-(10), we have

$$nT(r,f) \leq nN\left(r, \frac{1}{f}\right) + 4nN(r,f) + N\left(r, \frac{1}{\Phi(z) - z}\right) + S(r,f). \tag{11}$$

□

3. Proof of Theorems

Theorem 1.11.

Proof. Denoting $g = f - a$ by (11) we derive,

$$\begin{aligned} nT(r,f) \leq & nT(r,g) + O(1) \\ \leq & nN\left(r, \frac{1}{g}\right) + 4nN(r,g) + N\left(r, \frac{1}{\Phi(g) - z}\right) + S(r,g) \\ = & nN\left(r, \frac{1}{f-a}\right) + 4nN(r,f) + N\left(r, \frac{1}{\Phi(z) - z}\right) + S(r,f). \end{aligned} \tag{12}$$

Since $\delta(0,f) > 0$ and $\delta(\infty,f) = 1$, there is a positive number $\theta < 1$ such that

$$N\left(r, \frac{1}{f-a}\right) < \theta T(r,f) \tag{13}$$

$$N(r,f) = O(1)T(r,f). \tag{14}$$

If $\Phi(z)$ assumes only a finite number of fixed points, then from (12)-(14), we would have

$$n(1 - O(1) - \theta)T(r,f) \leq N\left(r, \frac{1}{\Phi(z) - z}\right), r \notin E, r \longrightarrow \infty, \tag{15}$$

where E is a possible exceptional set with finite logarithmic measure. Noticing f is transcendental, applying Lemma 2.6 and (15), we can get that $\Phi(z)$ assumes has infinitely many fixed points and $\tau(\Phi(z)) = \sigma(f)$. □

Theorem 1.12.

Proof. Since $\delta(0, f) = 1$ and $\delta(\infty, f) = 1$,

$$N\left(r, \frac{1}{f}\right) = S(r, f) \tag{16}$$

$$N(r, f) = S(r, f) \tag{17}$$

Since

$$T(r, \Phi(z)) \leq \sum_{i=1}^{n} T(r, f(z + c_i) + T(r, (f(z))^n). \tag{18}$$

Using Lemma 2.4, we can derive from (18) that

$$T(r, \Phi(z)) \leq 2nT(r, f) + S(r, f) \tag{19}$$

From (16)-(19), we have

$$\begin{aligned} nT(r, f) \leq & N\left(r, \frac{1}{\Phi(z) - z}\right) + S(r, f) \\ \leq & T(r, \Phi(z)) + S(r, f) \\ \leq & nT(r, f) + S(r, f) \end{aligned} \tag{20}$$

Since f is transcendental, (20) means that $M(f)$ has infinitely many fixed points and

$$T(r, M(f)) \sim nT(r, f) \sim N\left(r, \frac{1}{\Phi(z) - z}\right),$$

as $r \notin E, r \longrightarrow \infty$, where E is a possible exception set of r with finite logarithmic measure. □

4. Remark

If $m = n = 1$ and $a = 1$, then $\Phi(z)$ becomes the forward difference $\Delta_c f$, i.e.

$$\Phi = f(z + c) - f(z) = \Delta_c f(z).$$

Therefore, we can get the Theorems 1.9 and 1.10.

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Department of Mathematics, School of Engineering, Presidency University, Bangalore-560064, INDIA
E-mail address: jayjayaramhr@gmail.com

Department of Mathematics, Sri Mahaveera First Grade College, Moodbidri, Dakshina Kannada-574197, INDIA
E-mail address: harishjontyh@gmail.com

Department of Mathematics, School of Engineering, Presidency University, Bangalore-560064, INDIA
E-mail address: chinnuchaithra15@gmail.com

Department of Mathematics, School of Engineering, Presidency University, Bangalore-560064, INDIA
E-mail address: naveenkumarsh.220@gmail.com